

A COMBINATORIAL MODEL FOR REVERSIBLE RATIONAL MAPS OVER FINITE FIELDS

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ABSTRACT. We study time-reversal symmetry in dynamical systems with finite phase space, with applications to birational maps reduced over finite fields. For a polynomial automorphism with a single family of reversing symmetries, a universal (i.e., map-independent) distribution function $\mathcal{R}(x) = 1 - e^{-x}(1+x)$ has been conjectured to exist, for the normalized cycle lengths of the reduced map in the large field limit [20]. We show that these statistics correspond to those of a composition of two random involutions, having an appropriate number of fixed points. This model also explains the experimental observation that, asymptotically, almost all cycles are symmetrical, and that the probability of occurrence of repeated periods is governed by a Poisson law.

1. INTRODUCTION

This paper is concerned with time-reversal symmetry in dynamical systems with finite phase space, with applications to rational maps over finite fields. The concept of reversibility originated in the theory of smooth maps and flows on manifolds [5, 6, 4]. In this paper, a map L is said to be *reversible* if it is the composition of two involutions G and $H = L \circ G$. The involution G conjugates the map to its inverse, namely

$$(1) \quad G \circ L \circ G^{-1} = L^{-1}.$$

(For background information on reversibility and its generalization, see [11, 14, 18], and references therein.) The nature of the fixed sets of G and H , respectively $\text{Fix } G$ and $\text{Fix } H$, plays a key dynamical role in organizing the dynamics of a reversible map.

The reversibility property can be interpreted in purely algebraic terms, and in particular it applies to reversible algebraic maps L of any dimension n over any field K , namely reversible maps defined by rational functions with coefficients in K . If the field $K = \mathbb{F}_q$ is a *finite field* with q elements—where q is the power of a prime number p —we are led to the study of reversibility on a phase space with $N = q^n$ points. (For background reference on finite fields, see [13].) More meaningful is the case in which K is an algebraic number field, e.g., $K = \mathbb{Q}$. The same map L may then be reduced over infinitely many distinct finite fields, of increasing size.

We are concerned here with the case where the reductions of the component involutions G and H are involutory permutations of the phase space of $N = q^n$ points, so that the reduction of their composition L yields a permutation of this space. In reduction, the fixed sets $\text{Fix } G$ and $\text{Fix } H$ will now be finite. Examples arising from dimension $n = 2$ are the reversible planar polynomial automorphisms [7, 1, 8] which include the well-known area-preserving Hénon map $L_1 = H_1 \circ G_1$,

$$(2) \quad G_1 : (x, y) \mapsto (y, x) \quad H_1 : (x, y) \mapsto (x, -y + x^2 + a) \quad a \in K.$$

If $K = \mathbb{F}_p$, we find

$$\#Fix G_1 = \#Fix H_1 = p.$$

Reversible polynomial automorphisms in any dimension provide other examples. But certain rational maps in n dimensions can induce permutations, e.g., $L_2 = H_2 \circ G_2$ of [17]

$$\begin{aligned} G_2 : (x, y, z) &\mapsto (x + e(2y - k)(z + e(y - k)), k - y, z + e(2y - k)) & e, k \in K \\ H_2 : (x, y, z) &\mapsto (y(2 - 2x/F + x^2/F^2), x/F, -z) & F = 1 + (1 - y)^2. \end{aligned}$$

If $K = \mathbb{F}_p$, with $p \equiv 3 \pmod{4}$, the denominator in H_2 is non-vanishing and

$$\#Fix G_2 = p^2, \quad \#Fix H_2 = p.$$

The representability of G and H over a finite field \mathbb{F}_q imposes some restrictions on the parameters present in the involutions. For instance, if in (2), we take $a \in \mathbb{Q}$, say, we are led to consider the set P_a of the prime divisors of the denominator of a . Then the involution H_1 is representable over any finite field whose characteristic is not in P_a , such as the field of integers modulo a prime $p \notin P_a$. Additionally, one must verify that G and H remain involutions when reduced to a finite field. This is an instance of the property of *good reduction* (for a discussion of reduction in arithmetic dynamics, see [21, chapter 2]).

Representability assumed, the reduction of an algebraic map can be associated with interesting asymptotic phenomena, concerning the statistical behaviour of the periodic orbits in the large field limit. A main theme in this area of research is to ascertain whether such phenomena are *universal*, namely independent from the map, depending only on its structural properties e.g., reversibility. Furthermore, in reduction, the original dimension n of the underlying algebraic map ceases to be important as the finite phase space does not inherit the topology of the continuum \mathbb{R}^n or \mathbb{C}^n .

If the reduction of a reversible algebraic map is a permutation, it has been shown [20] that over finite fields, reversibility manifests itself combinatorially. In particular, the number of symmetric periodic orbits —those invariant under G — equals

$$(3) \quad (\#Fix G + \#Fix H)/2.$$

Strong experimental evidence [20, 9] suggests that the symmetric periodic orbits dominate the statistics over the asymmetric ones¹, so that (3) asymptotically counts the number of periodic orbits in the permutation. Furthermore, numerical experiments [20, 9] suggest there exists an asymptotic (large fields) distribution of the periods of the orbits, given by

$$(4) \quad \mathcal{R}(x) = 1 - e^{-x}(1 + x) \quad x \geq 0.$$

(In these experiments, ‘large fields’ refers to the limit \mathbb{F}_p , $p \rightarrow \infty$; there are other asymptotic regimes, see remark (3) in section 4.) The distribution (4) describes the limiting probability of the set of points belonging to cycles with scaled period not exceeding x . This distribution is believed to be universal, within the class of reversible maps with a single time-reversal symmetry. By contrast, maps which are not reversible appear to behave like random permutations [20]; the associated distribution, also conjectured to be universal, is markedly different from that of reversible maps.² Finally, there are asymptotic period distributions for integrable systems [19, 10]. Even though these distributions are map-dependent, they all feature a distinctive ‘quantization’ of periods. With appropriate scaling, the allowed periods occur

¹This is in sharp contrast with the case of maps with real or complex coordinates [14].

²The inadequacy of random permutations as a model for reversible maps on a discrete space was noted as far back as [16].

at the reciprocals of the natural numbers, resulting in step-like distribution functions. These various asymptotic phenomena have led to the development of simple and effective tests for detecting integrability and reversibility in algebraic mappings [19, 20, 10].

In this paper we propose a combinatorial model for the orbits, over a finite field, of a map with a single family of reversing symmetries. Our model consists of the composition of two random involutions. The size N of the space and the numbers $\#Fix G$ and $\#Fix H$ of fixed points of each involution are the parameters of the model. By letting N increase, we build a sequence of probability spaces. Under very general conditions, we derive the distribution (4) found in [20, 9].

We prove the following theorem, which appears in section 2 as theorems 1 and 7.

Theorem A. *Let (G, H) be a pair of random involutions of a set Ω with N points, and let $g = \#Fix G$ and $h = \#Fix H$. Let $\mathcal{R}_N(x)$ be the expectation value of the portion of Ω occupied by cycles of $H \circ G$ with period less than $2xN/(g+h)$, computed with respect to the uniform probability. If, with increasing N , g and h satisfy the conditions*

$$(5) \quad \lim_{N \rightarrow \infty} g(N) + h(N) = \infty \quad \lim_{N \rightarrow \infty} \frac{g(N) + h(N)}{N} = 0$$

then for all $x \geq 0$, we have the limit $\mathcal{R}_N(x) \rightarrow \mathcal{R}(x)$, the distribution function in (4). Moreover, almost all points in Ω belong to symmetric cycles.

This result is formulated in terms of the sequence $g+h$, and not of the two sequences individually. The conditions on the growth rate of $g+h$ are quite mild, hence the wide scope of applicability of the theorem. If g and h do not grow at the same rate, an interesting dynamical phenomenon occurs: asymptotically, almost all periodic orbits have even period (corollary 9). For n -dimensional maps over the finite field \mathbb{F}_q , one has $N = q^n$, while g and h will typically grow algebraically (see remark 1 of section 4). As mentioned above, increasing the dimension does not introduce new difficulties, because the absence of a significant topological structure makes the combinatorial model effectively dimension-independent. All that is needed is to keep track of the possible cardinalities of the fixed sets of the two involutions.

In [20], experimental evidence from reductions of the Hénon map (2) suggested that the probability of occurrence of several cycles with the same period followed a Poisson law, with parameter depending exponentially on the (rescaled) period. These findings motivate the study of the statistics of the occurrence of repeated periods in the random involutions model. In section 3 we consider this question, and prove the following result.

Theorem B. *With the notation of theorem A, let $\mu(t, i)$ be the probability that (G, H) has i cycles of period t . Assume that the sequence*

$$(6) \quad f(N) = \begin{cases} \frac{gh}{N} & \text{if } t \text{ is odd} \\ \frac{g^2 + h^2}{2N} & \text{if } t \text{ is even} \end{cases}$$

has the (possibly infinite) limit

$$c = \lim_{N \rightarrow \infty} f(N)$$

and let

$$(7) \quad y = \frac{(t-1)(g+h)}{2N} - \ln(f).$$

Then, if $c \neq 0$ we have, as $N \rightarrow \infty$

$$(8) \quad \mu(t, i) \sim e^{-\alpha} \frac{\alpha^i}{i!} \quad \alpha = \begin{cases} c & \text{if } t \text{ is constant} \\ e^{-y(t)} & \text{if } y \text{ is constant} \end{cases}$$

while if $c = 0$ we have $\mu(t, i) \sim \delta_i$, where δ is Kronecker's delta.

As it will appear in the proof of theorem B, in the case of constant y , for convergence to a Poisson distribution it is only required that $f(N) \not\rightarrow 0$, not the actual existence of a non-zero limit c . We conclude section 3 by showing the agreement between the prediction of theorem B with the actual repetitions $\mu(t, i)$ for a small t in the Hénon map. The latter are governed by the roots (mod p) of certain polynomial functions of one variable, with a connection to the Galois groups of these polynomials.

In section 4 we discuss the question of the asymptotic (large field) growth rate of the parameters associated with a rational map over a finite field, namely the numbers g and h of fixed points of the involutions. In this case N is the power of a prime number. We show that the growth conditions of theorem A on $g + h$ are easily satisfied, while $f(N)$ in theorem B depends algebraically on N , and all scenarios considered in the theorem can be realized. Finally, we also sketch how the random involution model might be extended to the case of reversible rational maps with singularities.

2. COMPOSITION OF RANDOM INVOLUTIONS

Let N be a positive integer, and let g, h be integers in the range $1 \leq g, h \leq N$ such that $N - g$ and $N - h$ are both even. We consider ordered pairs (G, H) of random involutions on a set Ω of N points, which fix g and h points, respectively. Thus $g = \#Fix G$, $h = \#Fix H$, and the cycle decomposition of G consists of g fixed points and $(N - g)/2$ two-cycles, and similarly for H . For each pair (G, H) of involutions we consider the composition $L = H \circ G$, which is a reversible permutation of Ω .

Let \mathbf{G} and \mathbf{H} be the sets of all involutions with the given parameters. We regard the space

$$(9) \quad \mathbf{E} = \mathbf{E}(g, h, N) = \mathbf{G} \times \mathbf{H}$$

as a probability space, with the uniform probability. We define the random variable $P_t : \mathbf{E} \rightarrow \mathbb{R}$ to be the fraction of the space Ω occupied by the t -cycles of the map $H \circ G$, namely

$$(10) \quad P_t = \frac{1}{N} \# \{x \in \Omega : x \text{ has minimal period } t \text{ under } H \circ G\} \quad t = 1, 2, \dots$$

For given N , the sequence (P_t) has only a finite number of non-zero terms.

The main object under study is the distribution function

$$(11) \quad \mathcal{R}_N(x) = \sum_{t=1}^{\lfloor xz \rfloor} \langle P_t \rangle \quad x \geq 0$$

where $\lfloor \cdot \rfloor$ is the floor (greatest integer part) function, z is a scaling parameter to be specified below, and the average $\langle \cdot \rangle$ is computed with respect to the uniform probability on \mathbf{E} (i.e., $\langle P_t \rangle = (\#\mathbf{E})^{-1} \sum_{\mathbf{E}} P_t$). If $\lfloor xz \rfloor = 0$, the sum is empty, and $\mathcal{R}_N(x)$ is defined to be zero.

The asymptotic parameter is N . We shall assume that $g = g(N)$ and $h = h(N)$ are positive integer sequences satisfying (5). These sequences determine the sequence of probability spaces

to be considered. To get non trivial asymptotics, the scaling parameter z must grow with N at an appropriate rate. To this end, we define the rational sequence

$$(12) \quad z(N) = \frac{2N}{g(N) + h(N)}$$

which, due to (5), diverges to infinity. The quantity $z(N)$ is the ratio of the size of the phase space to the number of symmetric cycles —cf. equation (3). Its full significance will be clarified at the end of this section (corollary 8).

The main purpose of this section is to prove theorem A of the introduction, which is an amalgamation of theorems 1 and 7.

Theorem 1. *The distribution function \mathcal{R}_N defined in (11), with scaling sequence $z(N)$ given by (12), admits the following limit*

$$\lim_{N \rightarrow \infty} \mathcal{R}_N(x) = \mathcal{R}(x) = 1 - e^{-x}(1+x) \quad x \geq 0.$$

It turns out that to establish theorem 1 it suffices to consider symmetric orbits only, because they provide the dominant contribution to the probability $\langle P_t \rangle$ in equation (11) — see theorem 7 below. Accordingly, we separate out the contribution deriving from symmetric and asymmetric cycles as follows

$$(13) \quad P_t = P_t^{(s)} + P_t^{(a)}.$$

We begin with symmetric cycles of odd period $t = 2k - 1$. It is known [4, 5, 11, 14, 18] that a cycle is of this type if and only if it corresponds to an orbit of L starting from $\text{Fix } G$ and reaching $\text{Fix } H$ for the first time after k iterations. Here and below, we follow the cycle dynamics keeping track of the action of G and H separately, which leads to a maximal sequence of $2k$ points

$$(14) \quad (x_1, y_1, \dots, x_k, y_k) \quad y_j = G(x_j), \quad x_{j+1} = H(y_j) \quad k \geq 1, \quad j = 1, \dots, k$$

with the property that all x_j are distinct. We call such a sequence a k -arc. In the case of odd period, we have the condition

$$(15) \quad x_1 \in \text{Fix } G, \quad y_k \in \text{Fix } H$$

(hence $y_1 = x_1$) together with the requirement that the sequence contains $t = 2k - 1$ distinct points (see figure 1). We must now determine the total number of k -arcs of this type generated by the elements of \mathbf{E} .

Throughout the paper, the underlined exponent will denote the falling factorial powers

$$(16) \quad n^{\underline{0}} = 1 \quad n^{\underline{a}} = n(n-1)(n-2) \cdots (n-a+1), \quad a \geq 1.$$

The information we require is contained in the following two lemmas.

Lemma 2. *Let $\mathbf{E} = \mathbf{E}(g, h, N)$ be as in equation (9). Then*

$$(17) \quad \# \mathbf{E}(g, h, N) = \frac{1}{2^{N-(g+h)/2}} \frac{(N!)^2}{g! h! ((N-g)/2)! ((N-h)/2)!}.$$

PROOF. The cardinality of the set of all involutions on N points, with g fixed points, is given by

$$\binom{N}{g} \frac{(N-g)!}{2^{(N-g)/2} ((N-g)/2)!}$$

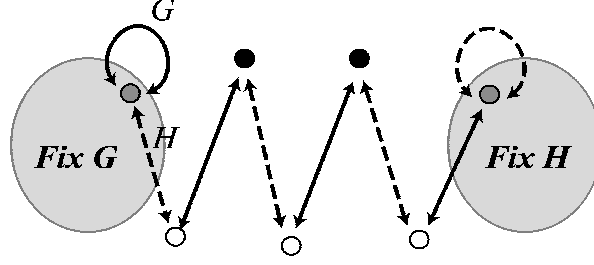


FIGURE 1. Construction of a 4-arc corresponding to a 7-cycle. The solid and dashed lines denote the action of G and H , respectively. The orbit of $L = H \circ G$ from $Fix G$ to $Fix H$ is represented by white circles; the black circles represent the return orbit from $Fix H$ to $Fix G$.

where the first term gives the number of ways of choosing the fixed points, and the second counts the number of ways of partitioning the remaining points into unordered pairs. Multiplying the expression above with the corresponding expression for h fixed points, we obtain

$$\#E(g, h, N) = \frac{1}{2^{N-(g+h)/2}} \frac{(N!)^2}{g! h! ((N-g)/2)! ((N-h)/2)!}$$

as desired. \square

Lemma 3. *Let $P_t^{(s)}$ be as above. The following holds*

$$(18) \quad \langle P_{2k-1}^{(s)} \rangle = \frac{2k-1}{N} N^{\underline{2k-1}} \mathcal{E}_{2k-1} \quad k \geq 1$$

where

$$(19) \quad \begin{aligned} \mathcal{E}_{2k-1} &= \frac{\#E(g-1, h-1, N-2k+1)}{\#E(g, h, N)} \\ &= \begin{cases} \frac{gh}{N^2} & \text{if } k = 1 \\ \frac{gh}{(N^{\underline{2k-1}})^2} \prod_{j=0}^{k-2} (N-g-2j)(N-h-2j) & \text{if } k > 1. \end{cases} \end{aligned}$$

PROOF. Let $(x_1, y_1, \dots, x_k, y_k)$ be the required k -arc (see equation (14) and figure 1). The initial point x_1 belongs to $Fix G$, so that $y_1 = x_1$. The first half of the cycle consists of the points x_1, \dots, x_k , the second half of the points y_k, \dots, y_2 , giving $2k-1$ distinct points in total. Hence the desired number of arcs is

$$N^{\underline{2k-1}}.$$

Each k -arc occurs with multiplicity determined by the unconstrained action of the pairs of involutions on the rest of the space, which is given by

$$\#E(g-1, h-1, N-2k+1).$$

Thus the average value of $P_{2k-1}^{(s)}$ is given by

$$\langle P_{2k-1}^{(s)} \rangle = \frac{2k-1}{N} N^{\frac{2k-1}{2}} \frac{\#\mathbf{E}(g-1, h-1, N-2k+1)}{\#\mathbf{E}(g, h, N)}.$$

Finally, the product formula for \mathcal{E}_{2k-1} is obtained from lemma 2 with a straightforward computation. \square

By analogy with equation (11), we define the distribution function $\mathcal{R}_N^{(s,o)}$ for odd symmetric cycles as

$$(20) \quad \mathcal{R}_N^{(s,o)}(x) = \sum_{k=1}^{\lfloor (xz+1)/2 \rfloor} \langle P_{2k-1}^{(s)} \rangle \quad x \geq 0.$$

We have

Theorem 4. *For any sequences g, h satisfying (5), with z as in (12), we have, as $N \rightarrow \infty$*

$$\mathcal{R}_N^{(s,o)}(x) \sim \frac{2gh}{(g+h)^2} \mathcal{R}(x) \quad x \geq 0.$$

Note that the limit distribution is symmetrical in g and h . To prove theorem 4 we need a lemma.

Lemma 5. *Let $g(N), h(N)$ be as in (5), let $m = m(N)$ be a positive integer sequence, and let*

$$\lambda = \lambda(N) = \left(1 - \frac{g}{N}\right) \left(1 - \frac{h}{N}\right) \quad S_m = \sum_{k=1}^m (2k-1) \lambda^{k-1}.$$

Then, as $N \rightarrow \infty$

$$(21) \quad S_m \sim \frac{2}{\kappa^2} [1 - e^{-m\kappa}(m\kappa + 1)] \quad \kappa = \frac{g+h}{N}.$$

PROOF. For fixed N, m and λ are fixed, and the quantity S_m is the sum of an arithmetico-geometric progression, with value

$$S_m = \frac{1}{(1-\lambda)^2} \left\{ 1 + \lambda - \lambda^m [2m(1-\lambda) + 1 + \lambda] \right\}.$$

As $N \rightarrow \infty$, we have, from (5)

$$\lambda \sim 1 \quad (1-\lambda) \sim \kappa \quad \ln(\lambda) \sim -\kappa$$

so we find

$$S_m \sim \frac{1}{\kappa^2} [2 - e^{m \ln(\lambda)} (2m\kappa + 2)] \sim \frac{2}{\kappa^2} [1 - e^{-m\kappa} (m\kappa + 1)]$$

as desired. \square

Proof of theorem 4. From lemma 3, we find, for $k \geq 1$

$$(22) \quad \begin{aligned} \langle P_{2k-1}^{(s)} \rangle &= \frac{gh(2k-1)}{N} \frac{\prod_{j=0}^{k-2} (N-g-2j)(N-h-2j)}{N^{\frac{2k-1}{2}}} \\ &= \frac{gh(2k-1)}{N(N-(2k-2))} \prod_{j=0}^{k-2} \left(1 - \frac{g-1}{N-(2j+1)}\right) \left(1 - \frac{h}{N-2j}\right). \end{aligned}$$

Now fix m , with $0 < 2m < N$, and define

$$(23) \quad \lambda_- = \left(1 - \frac{g}{N-m}\right) \left(1 - \frac{h}{N-m}\right) \quad \lambda_+ = \left(1 - \frac{g-1}{N}\right) \left(1 - \frac{h}{N}\right).$$

For all k such that $1 \leq 2k-2 \leq 2m$, we have the inequalities

$$(24) \quad \lambda_-^{k-1} < \prod_{j=0}^{k-2} \left(1 - \frac{g-1}{N-(2j+1)}\right) \left(1 - \frac{h}{N-2j}\right) < \lambda_+^{k-1}$$

which, together with (22), give

$$(25) \quad \frac{gh(2k-1)}{N^2} \lambda_-^{k-1} < \langle P_{2k-1}^{(s)} \rangle < \frac{gh(2k-1)}{N(N-m)} \lambda_+^{k-1}.$$

Summing over k , we obtain the bounds

$$(26) \quad \mathcal{R}_- \leq \sum_{k=1}^m \langle P_{2k-1}^{(s)} \rangle \leq \mathcal{R}_+,$$

where

$$\mathcal{R}_- = \frac{gh}{N^2} \sum_{k=1}^m (2k-1) \lambda_-^{k-1} \quad \mathcal{R}_+ = \frac{gh}{N(N-m)} \sum_{k=1}^m (2k-1) \lambda_+^{k-1}.$$

Let now $m = m(N)$ be a positive integer sequence such that

$$\lim_{N \rightarrow \infty} m(N) = \infty \quad m = o(N).$$

Then $\lambda_{\pm} \sim \lambda$ and applying lemma 5, we obtain, as $N \rightarrow \infty$

$$(27) \quad \mathcal{R}_- \sim \mathcal{R}_+ \sim \frac{gh}{N^2} S_m \sim \frac{2gh}{(g+h)^2} [1 - e^{-m\kappa}(m\kappa + 1)].$$

We now fix a non-negative real number x . For the convergence of the expression above, we specialize the sequence $m(N)$ as follows

$$(28) \quad m\kappa = x \quad \text{or} \quad m(N) = \lfloor xz(N)/2 \rfloor$$

where κ and z were defined in equations (21) and (12), respectively. Due to (5), $m(N) = o(N)$, and for all sufficiently large N , we have $0 < m(N) < N/2$. Hence (26) is valid, and our result now follows from (20) and (27), noting that $(xz(N) + 1)/2 \sim xz(N)/2$. \square

The sequence $m(N)$ given in (28) could be defined by the more general requirement that the sequence $m(N)\kappa(N)$ converge to a positive real number c , where the limiting cases $c = 0, \infty$ are excluded as they would lead to trivial distribution functions. We have set $c = 1$, for normalization. We remark that the choice of this constant does not affect the coefficient of the distribution function, which determines its limiting value for large arguments. This fact will become relevant in the proof of theorem 1 below.

We now turn to even symmetric cycles, of period $t = 2k$. Each cycle of this type corresponds to an orbit originating on $Fix G$ (or $Fix H$), and returning to it for the first time after k iterations of L . The arc associated to such a cycle (see equation (14)) is a $(k+1)$ -arc, with the property that x_1 and x_{k+1} are in $Fix G$. Therefore $x_1 = y_1$ and $x_{k+1} = y_{k+1}$, and the total number of arcs of this type is equal to $N^{2k}/2$, where the denominator 2 prevents arcs

beginning and ending on $\text{Fix } G$ from being counted twice. The denominator 2 cancels with the 2 in the period $2k$, so we obtain

$$(29) \quad \langle P_{2k,g}^{(s)} \rangle = \frac{k}{N} N^{2k} \mathcal{E}_{2k} \quad k \geq 1$$

where

$$(30) \quad \begin{aligned} \mathcal{E}_{2k} &= \frac{\#\mathbf{E}(g-2, h, N-2k)}{\#\mathbf{E}(g, h, N)} \\ &= \begin{cases} \frac{g(g-1)(N-h)}{N(N-1)^2} & \text{if } k=1 \\ \frac{g(g-1)}{N^{2k}} \prod_{j=0}^{k-2} (N-g-2j) \prod_{j=0}^{k-1} (N-h-2j) & \text{if } k>1. \end{cases} \end{aligned}$$

Putting together equation (29) and (30), we obtain, for $k \geq 2$

$$\langle P_{2k,g}^{(s)} \rangle = \frac{g(g-1)k}{N(N-(2k-1))} \prod_{j=0}^{k-2} \left(1 - \frac{g-1}{N-(2j+1)}\right) \prod_{j=0}^{k-1} \left(1 - \frac{h}{N-2j}\right)$$

to be compared with equation (22).

Exchanging g by h in the formulae above gives the twin quantity $P_{2k,h}^{(s)}$ for $2k$ -cycles with two points of $\text{Fix } H$. The distribution function $\mathcal{R}_N^{(s,e)}$ for even symmetric cycles is obtained by adding the contributions from the two fixed sets

$$(31) \quad \mathcal{R}_N^{(s,e)}(x) = \sum_{k=0}^{\lfloor xz/2 \rfloor} (\langle P_{2k,g}^{(s)} \rangle + \langle P_{2k,h}^{(s)} \rangle) \quad x \geq 0.$$

An analysis very similar to that of theorem 4 leads to the following result, whose proof we omit for the sake of brevity.

Theorem 6. *For any sequences g, h satisfying (5), with z as in (12), we have, as $N \rightarrow \infty$*

$$\mathcal{R}_N^{(s,e)}(x) \sim \frac{g^2 + h^2}{(g+h)^2} \mathcal{R}(x) \quad x \geq 0.$$

PROOF OF THEOREM 1. From theorems 4 and 6 we deduce that the probability distribution associated to symmetric cycles is given by

$$\mathcal{R}_N^{(s,o)}(x) + \mathcal{R}_N^{(s,e)}(x) \sim \frac{2gh + g^2 + h^2}{(g+h)^2} \mathcal{R}(x) = \mathcal{R}(x).$$

Because

$$\lim_{x \rightarrow \infty} \mathcal{R}(x) = 1$$

this distribution function is properly normalized (cf. remarks following the proof of theorem 4), and so it accounts for all cycles. \square

It is implicit from theorem 1 that, asymptotically, asymmetric cycles have zero probability. Asymmetric t -cycles of $L = H \circ G$ come in pairs, mapped to one another under G . They correspond to t -arcs of the type

$$(32) \quad (x_1, y_1, \dots, x_t, y_t) \quad y_j = G(x_j) \neq x_j, \quad x_{j+1} = H(y_j) \neq y_j, \quad j = 1, \dots, t$$

i.e., they have $2t$ distinct entries. We can now determine the decay rate of the probability of asymmetric cycles.

Theorem 7. *Consider a composition of random involutions on a set of N points, whose fixed sets satisfy (5). As $N \rightarrow \infty$, the measure of the set of points that belong to asymmetric cycles is asymptotic to $(g(N) + h(N))^{-1}$; in particular, almost all cycles are symmetric.*

PROOF. Recall from (13) that $P_t^{(a)}$ is the probability of finding an asymmetric t -cycle. We compute its average over \mathbf{E} :

$$\begin{aligned} \langle P_t^{(a)} \rangle &= \frac{N^{2t}}{N} \frac{\#\mathbf{E}(g, h, N - 2t)}{\#\mathbf{E}(g, h, N)} \\ (33) \quad &= \frac{1}{N} \prod_{j=0}^{t-1} \left(1 - \frac{g-1}{N - (2j+1)} \right) \left(1 - \frac{h}{N - 2j} \right). \end{aligned}$$

From the above discussion, the right-hand side of this equation involves counting t -arcs like (32). Note that the period of the orbit is not present as a factor. This is because it is absorbed by the possibility to create t -arcs that are one and the same from (32) by cyclic permutations of the $\{x_j, y_j\}$ pairs (t possibilities) and the switch $x_j \leftrightarrow y_j$ together with reversing the order of the switched pairs (2 possibilities).

Let $\mathcal{R}_N^{(a)}$ be the associated distribution function

$$\mathcal{R}_N^{(a)}(x) = \sum_{t=1}^{\lfloor xz \rfloor} \langle P_t^{(a)} \rangle.$$

We now proceed as in the proof of theorem, with λ_{\pm} and z given by (23) and (12), respectively. Summing the expression above over t in the appropriate range gives the bounds

$$\mathcal{R}_- \leq \mathcal{R}_N^{(a)}(x) \leq \mathcal{R}_+,$$

where

$$\mathcal{R}_- = \frac{1}{N} \sum_{t=1}^{\lfloor xz(N) \rfloor} \lambda_-^t \quad \mathcal{R}_+ = \frac{1}{N} \sum_{t=1}^{\lfloor xz(N) \rfloor} \lambda_+^t.$$

This time the bounds are geometric progressions; as $N \rightarrow \infty$, we find

$$\mathcal{R}_- \sim \mathcal{R}_+ \sim \frac{1 - e^{-x}}{g(N) + h(N)},$$

as desired. □

The theorems of this section have some immediate consequences. Consider the sequence $z(N)$, defined in (12). Its denominator $g(N) + h(N)$ is twice the number of symmetric cycles [20, proposition 1], and from theorem 1, asymptotically, the symmetric cycles have full measure. These considerations give the following characterization of $z(N)$.

Corollary 8. *Asymptotically, the sequence $z(N)$ defined in (12) represents the average cycle length of the map $H \circ G$.*

An immediate consequence of theorems 4 and 6 is the following.

Corollary 9. *Let g, h be as above. If $g(N) = o(h(N))$ or $h(N) = o(g(N))$, then almost all cycles of $H \circ G$ have even period.*

3. OCCURRENCE OF REPEATED PERIODS

This section first gives the proof of theorem B, stated in the introduction. The asymptotic relations (5) will be assumed to hold, together with the condition $h = O(g)$. The latter does not entail loss of generality, as we may always exchange g and h , which amounts to considering the inverse mapping L^{-1} . After the proof, we compare the results of theorem B with the mechanism that determines the repetition statistics for small period in the Hénon map (2).

PROOF OF THEOREM B. Let $\mu(t, i)$ be the probability that a randomly chosen $(G, H) \in \mathbf{E}$ has exactly i cycles of period t . We wish to determine the asymptotic behaviour of μ ; due to theorem 7, it will suffice to consider symmetric cycles only.

We introduce some notation. Let γ be a t -cycle in Ω , and let Γ be any set of disjoint t -cycles. We stipulate that the first point of each cycle belongs to some symmetry line; then there are

$$(34) \quad \frac{N!t}{i!}$$

ways of choosing such cycles. We define

$$(35) \quad E_\gamma = \{(G, H) \in \mathbf{E} : \gamma \text{ is a } t\text{-cycle of } (G, H)\} \quad E_\Gamma = \bigcap_{\gamma \in \Gamma} E_\gamma.$$

Note that if the cycles of Γ were not disjoint, then E_Γ would be empty.

Given Γ , with $\#\Gamma = i$, we are interested in the number of pairs in \mathbf{E} that support the cycles of Γ , and that do not have any t -cycle in the complement $\Omega \setminus \Gamma$. The parameters of the involutions acting on $\Omega \setminus \Gamma$ are determined by the type of t -cycles being considered (odd period symmetric, even period symmetric on $Fix G$, etc.). Accordingly, we'll consider the following sets of pairs of involutions

$$(36) \quad \begin{array}{ll} \mathbf{E}(g-i, h-i, N-it) & \text{symmetric, odd period} \\ \mathbf{E}(g-2i, h, N-it) & \text{symmetric, even period, on } Fix G \\ \mathbf{E}(g, h-2i, N-it) & \text{symmetric, even period, on } Fix H \end{array}$$

whose cardinality is given by lemma 2. Whenever we don't require specialization, we'll denote any of the above sets by the common symbol \mathbf{E}_i . The pairs of involutions in \mathbf{E}_i act on the space Ω_i with $N-it$ elements.

Let $\bar{\mathbf{E}}_i$ be the set of elements of \mathbf{E}_i that have no t -cycles at all. By the inclusion-exclusion principle, we find

$$(37) \quad \begin{aligned} \#\bar{\mathbf{E}}_i &= \# \left(\mathbf{E}_i \setminus \bigcup_{\gamma \in \Omega} E_\gamma \right) = \sum_{\Gamma} (-1)^{\#\Gamma} \#E_\Gamma \\ &= \sum_{n \geq 0} (-1)^n \sum_{\#\Gamma=n} \#E_\Gamma. \end{aligned}$$

The number of non-zero summands is clearly finite.

To compute the inner sum for fixed t and i , we first choose n distinct cycles, and this can be done in $(N-it)\frac{nt}{n!}$ ways. Then we let the involutions act freely on the remaining space,

which gives $\#\mathbf{E}_{i+n}$ possibilities. Thus

$$\sum_{\#\Gamma=n} \#E_\Gamma = \frac{(N-it)^{nt}}{n!} \#\mathbf{E}_{i+n}.$$

Considering the equations (34) and (37), we obtain

$$(38) \quad \mu(t, i) = \frac{N^{it}}{i!} \frac{\#\bar{\mathbf{E}}_i}{\#\mathbf{E}_0} = \frac{N^{it}}{i!} \sum_{n=0}^M (-1)^n \frac{(N-it)^{nt}}{n!} \frac{\#\mathbf{E}_{i+n}}{\#\mathbf{E}_0}$$

where $M = \lfloor N/t - i \rfloor$.

We specialize formula (38) to the case of odd period t , using the notation $\mu^{(o)}$. For this purpose, we consider the first expression \mathbf{E} in (36), to obtain

$$\mu^{(o)}(i, t) = \frac{N^{it}}{i!} \sum_{n=0}^M (-1)^n \frac{(N-it)^{nt}}{n!} \frac{\#\mathbf{E}(g - (i+n), h - (i+n), N - (i+n)t)}{\#\mathbf{E}(g, h, N)}.$$

Inserting the values of $\#\mathbf{E}$ from lemma 2, we obtain, after some manipulation

$$(39) \quad \mu^{(o)}(t, i) = \frac{1}{i!} \sum_{n=0}^M \frac{(-1)^n}{n!} \mathcal{A}^{(o)}(i, n) \mathcal{B}^{(o)}(i, n)$$

where

$$(40) \quad \mathcal{A}^{(o)}(i, n) = \frac{g^{i+n} h^{i+n}}{(N - (i+n)(t-1))^{i+n}}$$

$$(41) \quad \mathcal{B}^{(o)}(i, n) = \begin{cases} 1 & \text{if } t = 1 \\ \prod_{j=0}^{(i+n)(t-1)/2-1} \left(1 - \frac{g-1}{N-(2j+1)}\right) \left(1 - \frac{h}{N-2j}\right) & \text{if } t \geq 1. \end{cases}$$

With a similar argument we derive the probability $\mu^{(e)}$ for even period

$$(42) \quad \mu^{(e)}(t, i) = \frac{1}{i!} \sum_{n=0}^M \frac{(-1)^n}{n!} \mathcal{A}^{(e)}(i, n) \mathcal{B}^{(e)}(i, n)$$

where

$$(43) \quad \mathcal{A}^{(e)}(i, n) = \frac{1}{2} \frac{g^{2(i+n)} + h^{2(i+n)}}{(N - (i+n)(t-1))^{i+n}}$$

$$(44) \quad \mathcal{B}^{(e)}(i, n) = \prod_{j=0}^{(i+n)(t-2)/2-1} \left(1 - \frac{g-1}{N-(2j+1)}\right) \left(1 - \frac{h}{N-2j}\right) \\ \times \prod_{j=(i+n)(t-2)/2}^{(i+n)t/2-1} \left(1 - \frac{h}{N-j}\right).$$

As in the analysis of section 2, this expression takes into account cycles on both $Fix G$ and $Fix H$, while the factor of 2 compensates for the double-counting that results from the presence of two points on each symmetry line.

We now consider the asymptotic behaviour of $\mu(t, i)$, for fixed i . Let y be given by (7), which we rewrite as

$$(45) \quad y = x - \ln(f) \quad x(t) = \frac{(t-1)(g+h)}{2N}.$$

For the purpose of scaling, the period t will be allowed to vary with N .

Let us consider the expression

$$(46) \quad \Pi(a, g, h) = \left(1 - \frac{g}{N-a}\right)^{J/2} \left(1 - \frac{h}{N-a}\right)^{J/2} \quad J = (i+n)(t-1) \quad 0 \leq a < N,$$

which, for $a = o(N)$, provides the dominant contribution to the products $\mathcal{B}(i, n)$, in both cases. A straightforward calculation gives

$$(47) \quad \Pi(a, g, h) = e^{-(i+n)x} \left[1 + O\left(\frac{ngx}{N}\right) + O\left(\frac{ngx}{N}\right)\right].$$

In view of the bounds

$$\Pi(J, g, h) \leq \mathcal{B}(i, n) \leq \Pi(0, g-1, h)$$

we compute

$$\begin{aligned} \Pi(0, g-1, h) &= e^{-(i+n)x} \left[1 + O\left(\frac{ngx}{N}\right)\right] \\ \Pi(J, g, h) &= e^{-(i+n)x} \left[1 + O\left(\frac{n^2x^2}{g}\right) + O\left(\frac{ngx}{N}\right)\right]. \end{aligned}$$

We obtain

$$|\mathcal{B}(i, n) - e^{-(i+n)x}| \leq |\Pi(0, g-1, h) - \Pi(J, g, h)|$$

and hence

$$(48) \quad \mathcal{B}(i, n) = e^{-(i+n)x} \left[1 + O\left(\frac{ngx}{N}\right) + O\left(\frac{n^2x^2}{g}\right)\right].$$

This estimate is valid for both odd and even period.

Next we consider the products $\mathcal{A}(i, n)$ in (40) and (43). Recalling the definition of $f(N)$ given in equation (6), and the fact that $h = O(g)$, we have the following asymptotic behaviour

$$(49) \quad \mathcal{A}(0, 0) = 1 \quad \mathcal{A}(i, n) = f(N)^{i+n} \left[1 + O\left(\frac{xn^2}{g}\right)\right], \quad i+n > 0$$

We must consider two cases.

Case I: $f(N) \not\rightarrow 0$. From (48) and (49), we obtain, for both even and odd period

$$\begin{aligned} \mu(t, i) &= \frac{1}{i!} \sum_{n \geq 0} \frac{(-1)^n}{n!} e^{-(i+n)y} \left[1 + O\left(\frac{n^2x}{g}\right)\right] \left[1 + O\left(\frac{ngx}{N}\right) + O\left(\frac{n^2x^2}{g}\right)\right] \\ &= \frac{(e^{-y})^i}{i!} \sum_{n \geq 0} \frac{(-1)^n e^{-ny}}{n!} + \frac{(e^{-y})^i}{i!} \sum_{n \geq 0} \frac{(-1)^n}{n!} \left[O\left(\frac{n^2x}{g}\right) + O\left(\frac{ngx}{N}\right)\right] \\ (50) \quad &= \frac{(e^{-y})^i}{i!} \left[e^{-(e^{-y})} + O\left(\frac{x}{g}\right) + O\left(\frac{xg}{N}\right)\right] \end{aligned}$$

where we have used the identities

$$\sum_{n \geq 0} \frac{(-1)^n n^2}{n!} = 0 = O(1) \quad \sum_{n \geq 0} \frac{(-1)^n n}{n!} = -\frac{1}{e} = O(1).$$

Now we adjust t so that y is constant. This gives $x = O(\ln(f))$, and for both odd and even period the error terms in (50) vanish. We have convergence to a Poisson distribution

$$(51) \quad \mu(t, i) \sim e^{-\alpha} \frac{\alpha^i}{i!} \quad \alpha = e^{-y}.$$

Note that for the above result to hold, we have only required that $f(N) \not\rightarrow 0$, not the existence of a limit for $f(N)$.

If $f(N)$ tends to a non-zero finite limit c , there is no logarithmic term in the function $y(N)$ —cf. (45)—and it is possible to consider the probability $\mu(t, i)$ without scaling the period t . For fixed t , we have $x = O(g/N) \rightarrow 0$ and $y \rightarrow -\ln(c)$, and equation (50) gives

$$\mu(t, i) \sim e^{-c} \frac{c^i}{i!},$$

independent of t .

Case II: $f(N) \rightarrow 0$. From (49) we have $\mathcal{A}(i, n) \sim \delta_{i+n}$ (Kronecker's delta), and hence

$$\mu(t, i) \sim \delta_i.$$

The proof of theorem B is complete. □

We remark that theorem A can actually be derived from theorem B since the distribution $\mathcal{R}(x)$ follows from first principles as

$$(52) \quad \mathcal{R}(x) = \lim_{N \rightarrow \infty} \mathcal{R}_N(x) = \lim_{N \rightarrow \infty} \sum_{t=1}^{\lfloor xN \rfloor} \frac{t}{N} \sum_{i=1}^{\lfloor N/t \rfloor} i \mu(t, i),$$

equivalently $\langle P_t \rangle$ follows from

$$(53) \quad \langle P_t \rangle = \frac{t}{N} \sum_{i=1}^{\lfloor N/t \rfloor} i \mu(t, i).$$

Note that the variable x in (45) is, asymptotically, the variable x in $\mathcal{R}(x)$. Taking the result (51) gives

$$\sum_{i=1}^{\infty} i \mu(t, i) = \alpha e^{-\alpha} \sum_{i=1}^{\infty} \frac{\alpha^{(i-1)}}{(i-1)!} = \alpha = e^{-y}$$

Since $e^{-y} = f e^{-x}$ from (45), we have

$$(54) \quad \langle P_t \rangle \sim \frac{t}{N} f e^{-x}.$$

If $t = 2k - 1$ is odd, we substitute for f from (6) and x from (45) to see

$$(55) \quad \langle P_{2k-1} \rangle \sim \frac{gh(2k-1)}{N^2} e^{-(k-1)\kappa}.$$

Comparing this to (25) and noting there that $\lambda_{\pm}^{k-1} \sim \lambda^{k-1} = e^{(k-1)\ln(\lambda)} \sim e^{-(k-1)\kappa}$ shows that we will recover the result of theorem 4. A similar exercise on (54) for $t = 2k$ using the appropriate f from (6) yields the result of theorem 6 and hence the distribution $\mathcal{R}(x)$ upon

summation of odd and even contributions. We have chosen to give the independent proof of Theorem A in section 2 as it is simple and direct and bypasses the calculation of $\mu(t, i)$.

The applicability of Theorem B to algebraic maps over finite fields should be considered with care. For fixed t , the maximum number of t -cycles for an algebraic map is plainly finite, because the points in a cycle are solutions of a set of polynomial equations; therefore, in the large field limit, one cannot possibly expect convergence to (8). Under what circumstances is then theorem B a model for such maps?

To articulate this question, let us consider a specific example, namely the Hénon map (2), with $a = 1$. The set $Fix G$ is the line $y = x$, and so over the field \mathbb{F}_p , this map has parameters $g = h = p$, hence $\alpha = c = 1$ in theorem B. Let us consider symmetric cycles, which, due to theorem A, are expected to dominate the dynamics. When the fixed sets are one-dimensional, the study of symmetric orbits is greatly simplified, because the points in these orbits are roots of polynomial equations in one indeterminate (as opposed to being points on varieties). Now, any symmetric odd cycle has a point (x, x) on $Fix G$, and the values of x for the period $t = 5$ are found to be the roots of the irreducible polynomial

$$\Phi_5(x) = x^6 - 2x^5 + 5x^4 - 6x^3 + 8x^2 - 4x + 3.$$

Thus the number of symmetric 5-cycles over the field \mathbb{F}_p is equal to the number of linear factors in the factorization of Φ_5 modulo p . In particular, the maximum number of 5-cycles is equal to 6, the degree of Φ_5 .

In general, polynomial factorization over a finite field is a highly irregular process, subject only to a slight deterministic constraint —Stickelberger’s theorem— which determines the parity of the number of factors [2, section 4.8]. Significantly, this process is ruled by a probabilistic law —Cebotarev’s density theorem— which determines the probability associated with each factorization type (number and degree of the factors) in terms of properties of the Galois group of the polynomial [15, p. 129]. Specifically, for each factorization with distinct roots, the degree of the factors defines an additive partition of the degree of the polynomial. Each partition in turn identifies uniquely a conjugacy class of the Galois group, with the terms in the partition corresponding to cycle lengths in the cycle decompositions of the permutations in the class. The probability of a factorization type is then given by the relative size of the corresponding class in the Galois group. If the Galois group of Φ is the symmetric group, the largest possible one, then every factorization type does actually occur, and with positive density among all primes.

It can be verified that the Galois group of Φ_5 is in fact S_6 , the set of all permutations of $\deg(\Phi_5) = 6$ objects [3]. There are therefore 11 possible types of factorizations of Φ_5 , corresponding to the 11 additive partitions of the integer 6. We factor $\Phi_5(x)$ modulo a few consecutive primes

$$\Phi_5(x) \equiv \begin{cases} (x + 508)(x + 917)(x + 1165)(x + 1486)(x + 3168)(x + 5880) & (\text{mod } 6563) \\ x^6 + 6567x^5 + 5x^4 + 6563x^3 + 8x^2 + 6565x + 3 & (\text{mod } 6569) \\ (x + 1265)(x + 3889)(x^4 + 1415x^3 + 2999x^2 + 4200x + 2683) & (\text{mod } 6571). \end{cases}$$

These factorizations correspond to the partitions $1 + 1 + 1 + 1 + 1 + 1 = 6 = 1 + 1 + 4$, respectively. We see that, over the field \mathbb{F}_p , the Hénon map has six symmetric 5-cycles for $p = 6563$ (the smallest prime for which this happens), no 5-cycles at all for $p = 6569$, and two 5-cycles for $p = 6571$.

The degree of the polynomials Φ_t grows exponentially with t , and it is reasonable to expect that, typically, they will be irreducible, with large Galois groups. For the symmetric group,

the Cebotarev's probability $\nu(t, i)$ of occurrence of i symmetric cycles of period t is just the probability of having exactly i fixed points in the appropriate symmetric group, namely

$$(56) \quad \nu(t, i) = \frac{1}{i!} \sum_{j=0}^{d-i} (-1)^j \frac{1}{j!} \quad d = \deg \Phi_t(x).$$

In the large t limit, $\nu(t, i)$ converges to $e^{-1}/i!$, independent of t . This agrees with the prediction of the random involution model.

It is instructive to compare the data (56) from Cebotarev's theorem with the prediction of theorem B, for the period $t = 5$.

i	Cebotarev thm	thm B
0	$53/144 = 0.3681\dots$	$0.3679\dots$
1	$11/30 = 0.3667\dots$	$0.3679\dots$
2	$3/16 = 0.1875\dots$	$0.1839\dots$
3	$1/18 = 0.0555\dots$	$0.0613\dots$
4	$1/48 = 0.0208\dots$	$0.0153\dots$
5	0	$0.0031\dots$
6	$1/720 = 0.0014\dots$	$0.0005\dots$

We see that, even for such a low period, for $i < 3$ the agreement is already reasonable.

Thus, in the two-dimensional case, some statements on the probability of multiple occurrences of symmetric t -cycles translate into statements on the maximality of the Galois groups of Φ_t . Even though this formulation would allow one to re-cast some dynamical questions in algebraic terms, we note that problems related to Galois groups of iterated polynomials are often quite difficult.

4. CONCLUDING REMARKS

When the reduction over a finite field of a reversible algebraic map with a single family of reversing symmetries is a permutation, numerical experiments suggest the period distribution function is given by the period distribution function $\mathcal{R}(x)$ of (4). In this paper, we have shown that $\mathcal{R}(x)$ corresponds to the expected period distribution function in a probability space of pairs of involutions characterized by the cardinalities of their fixed sets. We find that the same probability space furnishes a Poisson law for the probability of cycles with the same period, also in agreement with numerics.

We conclude with some remarks:

(1) To prove theorem A, only mild regularity conditions —the existence of the limits (5) — were imposed on the behaviour of the sequences $g(N)$, $h(N)$, specifying the cardinalities of the fixed sets of the two involutions, as the size $N = q^n$ of the phase space increases (with n the dimensionality of the map). Similarly, theorem B involves checking the behaviour of the sequence $f(N)$ in (6). We point out that when a reversible map of a finite set is obtained by reducing an algebraic mapping to a finite field, the behaviour of such sequences is strongly constrained, resulting in algebraic growth.

A simple but significant case comprises maps for which the fixed set $Fix G$ is an affine subspace of some linear space. An example is given by two-dimensional reversible polynomial automorphisms over some field K , where the existence of a normal form for the two involutions

ensures that their fixed set is a single point or a line [1]. Localization to the finite field \mathbb{F}_q gives a phase space with $N = q^2$ points, and since a line in the affine plane \mathbb{F}_q^2 has q points, we have an algebraic growth: $g(N) + h(N) = 2q = 2\sqrt{N}$. In dimension d , each of $\text{Fix } G$ and $\text{Fix } H$ can have integer dimension ranging from 0 to $d - 1$, and if these sets are affine spaces, we obtain the sequence $g(N) + h(N) = N^r + N^s$, with $r, s \in \mathbb{Q}$, $0 \leq r, s < 1$. The conditions (5) are then met as long as one of r or s is nonzero, i.e. as long as one of $\text{Fix } G$ and $\text{Fix } H$ is at least one-dimensional. Furthermore, in this situation, we see from (6) that $f(N) = N^{r+s-1}$ or $f(N) = (N^{2r-1} + N^{2s-1})/2$. So either $f = 1$ (e.g., when $r = s = 1/2 \Leftrightarrow g = h = \sqrt{N}$) or $f \rightarrow \infty$ algebraically or $f \rightarrow 0$ algebraically.

More generally, the sets $\text{Fix } G$ and $\text{Fix } H$ are algebraic varieties over \mathbb{F}_q determined by n polynomial equations, and we are interested in determining the number of points on these varieties. The number M of points on an r -dimensional irreducible algebraic variety in the n -dimensional projective space of degree d over the field \mathbb{F}_q satisfies the Hasse-Weil bound [12]

$$(57) \quad |M - q^r| \leq (d - 1)(d - 2)q^{r-1/2} + Cq^{r-1}$$

where $C = C(n, r, d)$. For $r = 1$ we obtain a square root bound, and we can also replace $(d - 1)(d - 2)$ by the genus of the curve. The result (57) ensures that, in the large field limit, there is algebraic growth of the cardinality of fixed sets. So the discussion of the previous paragraph applies concerning the validity of the asymptotic conditions (5) and the behaviour of $f(N)$.

(2) Experiments show that algebraic maps without time-reversal symmetry behave quite differently from reversible ones, when represented on a finite field [20]. In the absence of reversibility, the cycle distribution is found to be that of a random permutation, given by the identity function on the unit interval. This distribution, obtained via the scaling $z(N) = N$, has led to a general conjecture [20, conjecture 2].

(3) When a general rational reversible map is reduced over finite fields, we expect that points where the denominators vanish (singularities) will occur. Restricted to its periodic points, the reduced map is still a permutation. Numerically, we find the distribution of periods is still governed by $\mathcal{R}(x)$, consistent with Theorem A [9, 22]. But nonperiodic orbits will also appear in the reduction of such a map. We are presently investigating the extension of the random involution model to explain various aspects of the dynamics of the reduction to finite fields of general rational reversible maps [22]. We are also extending our numerical experiments to cover the asymptotic regime \mathbb{F}_q , $q = p^k$, $k \rightarrow \infty$.

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